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On the Order of Polynomial Approximation for Closed Jordan Domains

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1. INTRODUCTION

Let K be a closed Jordan domain bounded by the closed Jordan curve Γ. By the Riemann mapping theorem there exists a unique meromorphic function

$$z = \psi(\zeta) = \rho\zeta + b_0 + b_1\zeta^{-1} + b_2\zeta^{-2} + \cdots,$$

which maps $|\zeta| > 1$ conformally onto the complement of K (ρ is the *transfinite diameter* of K). It is well known that $\psi(\zeta)$ admits a continuous one-to-one extension to $|\zeta| \geqslant 1$. The *Faber polynomials* $F_n(z) = z^n + \cdots$ associated with the set K are defined (for $z \in K$) by the expansion

$$\frac{\psi'(\zeta)}{\psi(\zeta) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{\zeta^{n+1}} \rho^{-n}.$$

The boundary Γ is said to be of *bounded rotation* [9] if it is rectifiable and if there exists a real 2π-periodic function $u(\theta)$ having the following properties:

(i) $u(\theta)$ is of bounded variation (this implies that the right- and left-hand limits $u(\theta+)$ and $u(\theta-)$ exist for every θ).

(ii) Γ has a right and left tangent at every point, and at the point $z = \psi(e^{i\theta})$ the angle between the positive real axis and the right (resp. left) tangent to Γ is equal to $u(\theta-)$ (resp. $u(\theta+)$). $V = \int_0^{2\pi} |du(\theta)|$ is the *total rotation* of Γ.

The class of closed Jordan domains whose boundary is of bounded rotation will be denoted by BR. In particular, every bounded closed convex set belongs to BR (except for one-point sets and line-segments).

We will denote the class of functions continuous on K and regular in the interior of K by $\Lambda(K)$. Every function $f \in \Lambda(K)$ can be associated with a formal expansion:

$$f(z) \sim \sum_{m=0}^{\infty} c_m F_m(z) \rho^{-m}, \tag{1.1}$$

the so-called *Faber expansion* of $f(z)$. The numbers

$$c_m = \frac{1}{2\pi} \int_0^{2\pi} f(\psi(e^{is}))\, e^{-ims}\, ds \qquad (m = 0, 1, 2, \ldots)$$

are the *Faber coefficients* of $f(z)$; it is important to observe that, at the same time, they also are the complex Fourier coefficients (for $m \geqslant 0$) of the function $f(\psi(e^{it}))$. Every summability method which, when applied to Fourier series, gives a well approximating trigonometric polynomial, can, in principle, be also applied to the Faber expansion (1.1) to give a polynomial approximation of the function $f(z)$ on the set K. We will use the de la Vallée Poussin sums of the Faber expansion, but any other similar summability method would be just as effective. The de la Vallée Poussin sums are polynomials of degree $(2n - 1)$ defined by the formula

$$T_{2n-1}(z) = \sum_{k=0}^{2n-1} \lambda_k^{(n)} c_k F_k(z) \rho^{-k},$$

where $\lambda_k^{(n)} = 1$ for $0 \leqslant k \leqslant n$, $\lambda_k^{(n)} = \dfrac{2n-k}{n}$ for $n \leqslant k \leqslant 2n-1$.

The estimates we shall obtain for $f(z) - T_{2n-1}(z)$ will enable us to estimate the *order of polynomial approximation*, i.e., to find good upper bounds for the quantity

$$\rho_n(f, K) = \inf_{p \in \Pi_n} \max_{z \in K} |f(z) - p(z)|,$$

where Π_n is the class of polynomials of degree n.

In the present paper we shall estimate $\rho_n(f, K)$ for the class of sets BR and the class of functions $\Lambda(K)$. Similar results have been obtained (for other classes of sets and by different methods), among others, by Al'per [1] and Dzyadik [4, 6]. While our results are, in a certain sense, more general, they do not imply (or are implied by) these earlier results.

2. Statement of Results

THEOREM 1. *Let K be a closed Jordan domain whose boundary Γ is of bounded rotation. Let $f(z) \in A(K)$. Suppose that the function $F(\theta) = f(e^{i\theta}))$ satisfies Dini's condition*

$$\int_0^\pi \frac{\omega(t)}{t}\, dt < \infty, \tag{2.1}$$

where $\omega(x) = \omega(F, x)$ is the modulus of continuity of F.
Let

$$\omega_1(x) = \int_0^x \frac{\omega(t)}{t}\, dt + \omega(x) \tag{2.2}$$

(it is easy to show that $\lim_{x\to 0} \omega_1(x) = 0$).
Then, uniformly for $z \in K$:

$$|f(z) - T_{2n-1}(z)| \leqslant A_0 \frac{V}{\pi}\, \omega_1\left(\frac{1}{n}\right) \tag{2.3}$$

where A is an absolute constant and V is the total rotation of Γ. Thus, for ρ_n we have the estimate

$$\rho_{2n-1}(f, K) \leqslant A_0 \frac{V}{\pi}\, \omega_1\left(\frac{1}{n}\right). \tag{2.4}$$

Let us observe that if $\Omega_f = \Omega(f, y)$ is the modulus of continuity of $f(z)$ on Γ (or on K), and $\Omega_\psi = \Omega(\psi, x)$ is the modulus of continuity of $\psi(e^{i\theta})$, then trivially,

$$\omega(x) \leqslant \Omega_f(\Omega_\psi(x)). \tag{2.5}$$

COROLLARY 1. *If $\omega(x)$ is a "typical modulus of continuity"*, i.e., *such that, for some $q > 1$, and some $\epsilon > 0$:*

$$q^\epsilon \leqslant \frac{\omega(qx)}{\omega(x)} \tag{2.6}$$

for every x, then (see Lemma 5.1 below):

$$\omega_1(x) \leqslant C\omega(x). \qquad (C = C(\epsilon, q)). \tag{2.7}$$

Hence, if $\omega(x) = \omega(F, x)$ is a 'typical modulus', (2.3) and (2.4) can be replaced by

$$\rho_n(f, K) \leqslant \max_{z\in K} |f(z) - T_{2n-1}(z)| \leqslant C_1 \frac{V}{\pi}\, \omega\left(\frac{1}{n}\right) \leqslant C_1 \frac{V}{\pi}\, \Omega_f\left(\Omega_\psi\left(\frac{1}{n}\right)\right) \tag{2.8}$$

($C_1 = A_0C$). This upper bound is substantially the best possible (cf. [3]).

N.B. (2.8) remains true (suppressing the link $C_1 V/\pi\, \omega(1/n)$) if, instead of assuming that $\omega = \omega(F, x)$ satisfies (2.6) for some q, we assume that both Ω_f and Ω_ψ satisfy (2.6) for every q. The justification is immediate.

From Theorem 1, we can derive the following two results as special cases:

THEOREM 2. *Let K be a bounded closed convex set. Let $f(z) \in A(K)$. Suppose that $\Omega(x) = \Omega_f(x) = \Omega(f, x)$ satisfies Dini's condition* (2.1). *Let*

$$\Omega_1(x) = \int_0^x \frac{\Omega(t)}{t}\, dt + \Omega(x). \tag{2.9}$$

Then, if ρ is the transfinite diameter of K,

$$\rho_n(f, K) \leqslant \max_{z \in K} |f(z) - T_{2n-1}(z)| \leqslant 2\, A_0 \Omega_1\left(\frac{2\rho}{n}\right) \tag{2.10}$$

where A_0 is the absolute constant in (2.3).

COROLLARY 2 (see Lemmas 5.1 and 5.2). *If, in addition, we assume that Ω_f is a 'typical modulus'* (*cf.* *Corollary* 1), *then $\Omega_1(x) \leqslant C\Omega(x)$, and hence*

$$\rho_n(f, K) \leqslant \max_{z \in K} |f(z) - T_{2n-1}(z)| \leqslant C_2 \Omega_f\left(\frac{2\rho}{n}\right). \tag{2.11}$$

(2.11) *is clearly best possible, even for $K = \{z \mid |z| \leqslant \rho\}$* (*cf.* [2].).

We shall say that the closed Jordan curve Γ is *piecewise convex* if it is made up of a finite number of convex arcs (i.e., ones which are convex from the "inside" of Γ). Every piecewise convex curve is of bounded rotation.

THEOREM 3. *Let K be a closed Jordan domain whose boundary Γ is piecewise convex without any* 0 *external angles. Let $f(z) \in A(K)$. Suppose that $\Omega = \Omega_f(x) = \Omega(f, x)$ satisfies Dini's condition* (2.1). *Then, if $\pi\alpha$ $(0 < \alpha < 1)$ is the smallest external angle* (*the case*: $\alpha \geqslant 1$ *is covered by Theorem* 2), *we have*

$$\rho_n(f, K) \leqslant \max_{z \in K} |f(z) - T_{2n-1}(z)| \leqslant C_3 \Omega_1\left(\frac{\rho}{n^\alpha}\right), \tag{2.12}$$

where the constant C_3 depends on K only, and Ω_1 is defined by (2.9).

COROLLARY 3 (see Lemma 5.1). *If, in addition, we assume that Ω_f is a 'typical modulus', then $\Omega_1(x) \leqslant C\Omega(x)$, and hence*:

$$\rho_n(f, K) \leqslant \max_{z \in K} |f(z) - T_{2n-1}(z)| \leqslant C_4 \Omega_f\left(\frac{\rho}{n^\alpha}\right) \tag{2.13}$$

(2.13) *is best possible* (*cf.* [5]).

Remark. If, in (2.12) α is replaced by $\alpha - \epsilon$, the conclusion holds for every $\Gamma \in \mathrm{BR}$ which has no 0 external angles. (To verify this, compare Lemma 3.2 and [8, Lemma 6]).

3. Proof of Theorem 1

We may assume (without loss of generality) that $\rho = 1$.

By the assumption that $K \in \mathrm{BR}$, we have the following representation [10, Lemma 1] for the Faber polynomials:

$$F_k(\psi(e^{i\theta})) = \frac{1}{\pi} \int_0^{2\pi} e^{iks}\, d_s v(s, \theta), \tag{3.1}$$

where

$$v(s, \theta) = \arg(\psi(e^{is}) - \psi(e^{i\theta}))$$

$v(s, \theta)$ is a function of bounded variation, and [11, p. 1133]

$$\int_0^{2\pi} |\, d_s v(s, \theta)| \leqslant V, \tag{3.2}$$

where, as before, V is the total rotation of Γ.

Let $\sum_{k=-\infty}^{+\infty} c_k e^{ik\theta}$ be the complex Fourier series of $F(\theta) = f(\psi(e^{i\theta}))$ and let $\sum_{k=-\infty}^{+\infty} \tilde{c}_k e^{ik\theta}$ ($\tilde{c}_k = -ic_k$ for $k > 0, = +ic_k$ for $k < 0$, $\tilde{c}_0 = 0$) be its conjugate trigonometric series.

Applying (3.1), we obtain the representation

$$\begin{aligned} T_{2n-1}(\psi(e^{i\varphi})) &= \sum_{k=0}^{2n-1} \lambda_k^{(n)} c_k F_k(\psi(e^{i\varphi})) \\ &= \sum_{k=0}^{2n-1} \lambda_k^{(n)} c_k \frac{1}{\pi} \int_0^{2\pi} e^{ikt}\, d_t v(t, \varphi) \\ &= \int_0^{2\pi} \left(\sum_{k=0}^{2n-1} \lambda_k^{(n)} c_k e^{ikt} \right) d_t v(t, \varphi) \\ &= \frac{1}{\pi} \int_0^{2\pi} \left\{ \frac{1}{2} c_0 + \frac{1}{2} \sum_{k=-(2n-1)}^{2n-1} \lambda_{|k|}^{(n)} c_k e^{ikt} + \frac{1}{2} \sum_{k=-(2n-1)}^{2n-1} \lambda_{|k|}^{(n)} \tilde{c}_k e^{ikt} \right\} d_t v(t, \varphi). \end{aligned}$$

Let $\tilde{F}(\theta)$ denote the conjugate function of $F(\theta)$; it follows from the assumption (2.1) that $\tilde{F}(\theta)$ exists and is continuous. We write

$$F^*(\theta) = \tfrac{1}{2} c_0 + \tfrac{1}{2}(F(\theta) + \tilde{F}(\theta)).$$

Then,

$$\left| T_{2n-1}(\psi(e^{i\varphi})) - \frac{1}{\pi}\int_0^{2\pi} F^*(t)\, d_t v(t, \varphi) \right|$$

$$= \left| \frac{1}{2\pi}\int_0^{2\pi} \left\{ \sum_{k=-(2n-1)}^{2n-1} \lambda_{|k|}^{(n)} c_k e^{ikt} - F(t) \right\} d_t v(t, \varphi) \right.$$

$$\left. + \frac{1}{2\pi}\int_0^{2\pi} \left\{ \sum_{k=-(2n-1)}^{2n-1} \lambda_{|k|}^{(n)} \tilde{c}_k e^{ikt} - \tilde{F}(t) \right\} d_t v(t, \varphi) \right|$$

$$\leqslant \frac{1}{2\pi}\int_0^{2\pi} \left| \sum_{k=-(2n-1)}^{2n-1} \lambda_{|k|}^{(n)} c_k e^{ikt} - F(t) \right| | d_t v(t, \varphi)|$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} \left| \sum_{k=-(2n-1)}^{2n-1} \lambda_{|k|}^{(n)} \tilde{c}_k e^{ikt} - \tilde{F}(t) \right| | d_t v(t, \varphi)|$$

$$= \frac{1}{2\pi}\int_0^{2\pi} | \tau_{2n-1}(t) - F(t)| \, | d_t v(t, \varphi)|$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} | \tilde{\tau}_{2n-1}(t) - \tilde{F}(t)| \, | d_t v(t, \varphi)|. \tag{3.3}$$

Here,

$$\tau_{2n-1}(t) = \sum_{k=-2n+1}^{2n-1} \lambda_k^{(n)} c_k e^{ikt}$$

and

$$\tilde{\tau}_{2n-1}(t) = \sum_{k=-2n+1}^{2n-1} \lambda_k^{(n)} c_k e^{ikt}$$

are the de la Vallée Poussin sums of $F(\theta)$ and $\tilde{F}(\theta)$, respectively. It is well known [3] that

$$| \tau_{2n-1}(t) - F(t)| \leqslant A\rho_n^*(F)$$

and

$$| \tilde{\tau}_{2n-1}(t) - \tilde{F}(t)| \leqslant A\rho_n^*(\tilde{F}),$$

where $\rho_n^*(F)$ and $\rho_n^*(\tilde{F})$ are the degrees of best approximations of F, respectively $\tilde{F}$, by means of trigonometric polynomials, and A is an absolute constant. Thus, in view of (3.2), we obtain from (3.3) the inequality

$$\begin{aligned} &\left| T_{2n-1}(\psi(e^{i\phi})) - \frac{1}{\pi}\int_0^{2\pi} F^*(t)\, d_t v(t, \phi) \right| \\ &\leqslant \tfrac{1}{2}A\{\rho_n{}^*(F) + \rho_n{}^*(F)\}\frac{1}{\pi}\int_0^{2\pi} |\, d_t v(t, \phi)| \\ &\leqslant \frac{A}{2}\frac{V}{\pi}\{\rho_n{}^*(F) + \rho_n{}^*(\tilde{F})\}. \end{aligned} \tag{3.4}$$

It follows from the assumption (2.1) that the Faber series $\sum_{k=0}^{\infty} c_k F_k(z)$ converges uniformly to $f(z)$ [5, Theorem 5; 8, p. 54]. Hence, the same holds for the de la Vallée Poussin sums, and therefore, as a consequence of (3.4), we must have

$$\frac{1}{\pi}\int_0^{2\pi} F^*(t)\, d_t v(t, \phi) = f(\psi(e^{i\phi})) = F(\phi). \tag{3.5}$$

Substituting (3.5) into (3.4), we obtain

$$|\, T_{2n-1}(\psi(e^{i\phi})) - f(\psi(e^{i\phi}))| < \frac{A}{2}\frac{V}{\pi}\{\rho_n{}^*(F) + \rho_n{}^*(\tilde{F})\}.$$

Substituting: $z = \psi(e^{i\phi})$:

$$|\, T_{2n-1}(z) - f(z)| < \frac{A}{2}\frac{V}{\pi}\{\rho_n{}^*(F) + \rho_n{}^*(\tilde{F})\} \tag{3.6}$$

for all $z \in \Gamma$. It follows now from the maximum principle that (3.6) holds in fact for all $z \in K$.

It is known [12, 5.9.2] that

$$\rho_n{}^*(\tilde{F}) < C\left\{\rho_n{}^*(F) + \sum_{\nu=n+1}^{\infty}\frac{1}{\nu}\rho_\nu{}^*(F)\right\}$$

and hence ($C \geqslant 1$)

$$|\, T_{2n-1}(z) - f(z)| < AC\frac{V}{\pi}\left\{\rho_n{}^*(F) + \frac{1}{2}\sum_{\nu=n+1}^{\infty}\frac{1}{\nu}\rho_\nu{}^*(F)\right\}. \tag{3.7}$$

By Jackson's Theorem,

$$\rho_n{}^*(F) < B\omega\left(F, \frac{1}{n}\right). \tag{3.8}$$

Substituting (3.8) into (3.7), and making use of the elementary inequality

$$\frac{\omega(t_2)}{t_2} \leqslant 2\,\frac{\omega(t_1)}{t_1} \qquad \text{for} \quad t_1 < t_2$$

(cf. [12, 3.2.4]), we obtain that

$$\begin{aligned}
|\, T_{2n-1}(z) - f(z)| &< BAC\,\frac{V}{\pi}\left\{\omega\left(\frac{1}{n}\right) + \frac{1}{2}\sum_{\nu=n+1}^{\infty} \frac{1}{\nu}\,\omega\left(\frac{1}{\nu}\right)\right\} \\
&\leqslant ABC\,\frac{V}{\pi}\left\{\omega\left(\frac{1}{n}\right) + \sum_{\nu=n+1}^{\infty} \nu\omega\left(\frac{1}{\nu}\right)\frac{1}{\nu(\nu+1)}\right\} \\
&\leqslant ABC\,\frac{V}{\pi}\left\{\omega\left(\frac{1}{n}\right) + \sum_{\nu=n+1}^{\infty} \int_{1/\nu+1}^{1/\nu} \frac{\omega(t)}{t}\,dt\right\} \\
&\leqslant ABC\,\frac{V}{\pi}\left\{\omega\left(\frac{1}{n}\right) + \int_{0}^{1/n} \frac{\omega(t)}{t}\,dt\right\}
\end{aligned}$$

which proves (2.3).

4. Proofs of Theorems 2 and 3

Proof of Theorem 2. By a well-known result [5, p. 195]: $|\,\psi'(\zeta)| \leqslant 2\rho$ for $|\,\zeta\,| \geqslant 1$ (ρ is the transfinite diameter of K). Consequently, $\Omega_\psi(x) \leqslant 2\rho x$, and hence

$$\omega(x) = \omega(F, x) \leqslant \Omega_f(\Omega_\psi(x)) \leqslant \Omega_f(2\rho x).$$

Hence, observing that $V = 2\pi$ for convex curves, and applying (2.3),[1] we obtain that

$$\begin{aligned}
|f(z) - T_{2n-1}(z)| &\leqslant 2A_0\left\{\Omega_f\left(\frac{2\rho}{n}\right) + \int_0^{1/n} \frac{\Omega_f(2\rho t)}{t}\,dt\right\} \\
&= 2A_0\Omega_1\left(\frac{2\rho}{n}\right).
\end{aligned}$$

Thus (2.10) has been established.

Before proceeding to the proof of Theorem 3, we formulate a few lemmas. It will be assumed throughout (without loss of generality) that $\rho = 1$.

[1] In the special case when K is a line-segment, (2.3), strictly speaking, is not applicable, since the boundary of K is not a Jordan curve. However, *all* the results remain valid in this case.

LEMMA 4.1. [9]. *If Γ is of bounded rotation, the derivative of the mapping function $\psi(\zeta)$ has the following integral representation*:

$$\log \psi'(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \log\left(1 - \frac{e^{i\vartheta}}{\zeta}\right) du(\vartheta)$$

(*here* $u(\vartheta)$ has the same meaning as in the introduction).

LEMMA 4.2. *If Γ is piecewise convex, and its smallest exterior angle is $\alpha\pi$ $(0 \leqslant \alpha \leqslant 1)$, then,*

$$|\psi'(\zeta)| \leqslant \frac{C}{\left(1 - \frac{1}{|\zeta|}\right)^{1-\alpha}}. \tag{4.1}$$

Proof. Since $u(\vartheta)$ is of bounded variation, we can write

$$u(\vartheta) = u^+(\vartheta) - u^-(\vartheta),$$

$$V = \int_0^{2\pi} |du(\vartheta)| = \int_0^{2\pi} du^+(\vartheta) + \int_0^{2\pi} du^-(\vartheta),$$

where $u^+(\vartheta)$ and $u^-(\vartheta)$ are increasing functions. Suppose that the vertices of Γ are at the points

$$z_k = \psi(e^{i\vartheta_k}) \qquad (0 \leqslant \vartheta_1 < \vartheta_2 < \cdots < \vartheta_n < 2\pi),$$

and that the exterior angle at z_k is $\pi\alpha_k$. Since Γ is piecewise convex, $u^-(\vartheta)$ is a step-function; in fact,

$$u^-(\vartheta) = \sum_{\vartheta_k < \vartheta} (1 - \alpha_k).$$

Hence, applying Lemma 4.1,

$$\log \psi'(\zeta) = \frac{1}{\pi} \int_0^{2\pi} \log\left(1 - \frac{e^{i\vartheta}}{\zeta}\right) du^+(\vartheta) - \sum_{k=1}^{n} (1 - \alpha_k) \log\left(1 - \frac{\zeta_k}{\zeta}\right),$$

where $\zeta_k = e^{i\vartheta k}$. Hence, denoting the first integral by $g(\zeta)$:

$$\psi'(\zeta) = e^{g(\zeta)} \prod_{k=1}^{n} \left(1 - \frac{\zeta_k}{\zeta}\right)^{\alpha_k - 1}. \tag{4.2}$$

For $g(\zeta)$ we have the estimate

$$\operatorname{Re} g(\zeta) = \frac{1}{\pi} \int_0^{2\pi} \log \left| 1 - \frac{e^{i\vartheta}}{\zeta} \right| du^+(\vartheta)$$
$$\leqslant \frac{\log 2}{\pi} \int_0^{2\pi} du^+(\vartheta) \leqslant \frac{\log 2}{\pi} V.$$

Let

$$\arg \zeta = \vartheta, \qquad \min_{k \neq i} | \vartheta_k - \vartheta_i | = 2\delta, \qquad \min_i | \vartheta - \vartheta_i | = | \vartheta - \vartheta_j |.$$

Then

$$| \vartheta - \vartheta_k | \geqslant \delta \qquad \text{for} \quad k \neq j,$$

and hence

$$\left| 1 - \frac{\zeta_k}{\zeta} \right| \geqslant \sin \delta.$$

Thus, from (4.2) we obtain the estimate

$$| \psi'(\zeta)| \leqslant 2^{V/\pi} \left(1 - \frac{1}{| \zeta |}\right)^{\alpha_j - 1} (\sin \delta)^{\Sigma_{k \neq j} (\alpha_k - 1)} = \frac{A_j}{\left(1 - \frac{1}{| \zeta |}\right)^{1-\alpha_j}} .$$

Since $\alpha = \min_j \alpha_j$, we obtain (4.1) with $B = \max_j A_j$.

LEMMA 4.3 (Hardy–Littlewood, [7, p. 361]). *If $\psi(\zeta)$ is regular for $| \zeta | > 1$, continuous for $| \zeta | \geqslant 1$, and*

$$| \psi'(\zeta)| \leqslant \frac{C}{\left(1 - \frac{1}{| \zeta |}\right)^{1-\alpha}} \qquad (0 < \alpha < 1),$$

then $\psi(\zeta)$ satisfies a Lipschitz condition with exponent α on $| \zeta | = 1$, i.e.,

$$\Omega_\psi(x) \leqslant Bx^\alpha.$$

LEMMA 4.4. *Every closed Jordan curve Γ satisfying the conditions of Theorem 3, has the following property:*

If z_1 and z_2 are points of K, then there exists a rectifiable path γ in K, joining z_1 and z_2, such that

$$\text{length of } \gamma \leqslant \mu \, | z_1 - z_2 |,$$

where μ depends only on K.

We omit the proof of this lemma; the reader will have no difficulty in supplying it.

LEMMA 4.5. *For every closed Jordan domain K satisfying the conditions of Theorem* 3, *the inequality*

$$\Omega_\psi(mx) \leqslant m\mu\Omega_\psi(x) \tag{4.3}$$

holds for every $x > 0$, *and every positive integer* m. *The constant* μ *depends only on* K.

This lemma is an easy consequence of the previous one.

COROLLARY. For every real $r > 1$,

$$\Omega_\psi(rx) \leqslant (r+1)\,\mu\Omega_\psi(x) \tag{4.4}$$

Proof of Theorem 3 (we assume that $\rho = 1$). Applying Lemmas 4.2 and 4.3, we conclude that

$$\Omega_\psi(x) \leqslant Bx^\alpha.$$

Hence, applying (2.5) and (4.4),

$$\omega(x) = \omega(F, x) \leqslant \Omega_f(Bx^\alpha) \leqslant (B+1)\,\mu\Omega_f(x^\alpha), \tag{4.5}$$

and

$$\omega_1(x) \leqslant (B+1)\mu \int_0^x \frac{\Omega(t^\alpha)}{t}\,dt + (B+1)\,\mu\Omega_f(x^\alpha).$$

Substituting $s = t^\alpha$,

$$\omega_1(x) \leqslant (B+1)\mu \left\{\int_0^{x^\alpha} \frac{\Omega(s)}{s}\,ds + \Omega(x^\alpha)\right\}. \tag{4.6}$$

(2.13) is now an immediate consequence of (2.3), (4.6) and (2.9).

5. STATEMENT AND PROOF OF LEMMA 5.1

LEMMA 5.1. *If* $\omega(x)$ *is an increasing function for* $0 \leqslant x \leqslant h$, $\omega(0) = 0$, *and if for some* $q > 1$ *and some* $\epsilon > 0$,

$$\omega(qx)/\omega(x) \geqslant q^\epsilon \qquad \text{for every} \quad 0 < x \leqslant h/q,$$

then

$$\int_0^x \frac{\omega(t)}{t}\,dt \leqslant \frac{\log q}{1 - q^{-\epsilon}}\,\omega(x) \qquad \text{for} \quad 0 < x \leqslant h.$$

Proof.

$$\int_0^x \frac{\omega(t)}{t}\,dt = \sum_{n=0}^{\infty} \int_{q^{-n-1}x}^{q^{-n}x} \frac{\omega(t)}{t}\,dt = \sum_{n=0}^{\infty} \int_{x/q}^{x} \frac{\omega(q^{-n}s)}{s}\,ds$$

$$\leqslant \sum_{n=0}^{\infty} \omega(q^{-n}x) \int_{x/q}^{x} \frac{ds}{s} = \log q \sum_{n=0}^{\infty} \omega(q^{-n}x)$$

$$\leqslant \log q \cdot \omega(x) \sum_{n=0}^{\infty} q^{-n\epsilon} = \frac{\log q}{1 - q^{-\epsilon}}\,\omega(x).$$

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